Symplectic representation of a braid group on 3-sheeted covers of the Riemann sphere

Rolf-Peter Holzapfel

Preface

Why we called the class of two-dimensional Shimura varieties, which are not Hilbert modular, "Picard modular surfaces" ? In the mean time the name has been generally accepted, see e.g. Langlands (and others) [L-R]. On the one hand Picard worked on special Fuchsian systems of differential equations; on the other hand Shimura [Shi] introduced and investigated moduli spaces of abelian varieties with prescribed division algebra of endomorphisms, which are called (complex) "Shimura varieties" after some work of Deligne. One needs a chain of conclusions in a special case in order to connect both works. Picard found ad hoc on certain Riemann surfaces ordered sets of cycles, which we will call "Picard cycles" below. Quotients of integrals along these cycles solve (completely) a special Fuchsian system of differential equations. The basic solution consists of two multivalued complex functions of two variables. The multivalence can be described by the monodromy group of the system. By Picard-Lefschetz theory, actually described in Arnold (and others) [AVH], the monodromy group acts on the homology of an algebraic curve family respecting Picard cycles. In [H 95] (Lemma 2.27) we announced that the action on Picard cycles is transitive and, moreover, coincides with the action of an arithmetic unitary group $\mathbb{U}((2,1),\mathfrak{O}), \mathfrak{O}$ the ring of integers of an imaginary quadratic number field K. This is a key result. Namely, the unitary group is the modular group of the Shimura surface of (principally polarized) abelian threefolds with K-multiplication of type $(2,1)$. It parametrizes via Jacobians the isomorphy classes of the Riemann surfaces Picard started with. The aim of this article is to give a complete proof of the mentioned key result. It joins some actual and old mathematics. As a consequence one gets a solution of the relative Schottky problem for smooth Galois coverings of $\mathbb{P}^1(\mathbb{C})$ (Riemann sphere) of degree 3 and genus 3.

1 Basic facts, notations and definitions

We consider smooth compact complex curves (Riemann surfaces) C of genus 3 which are three-sheeted Galois coverings of the projective line (Riemann sphere) $\mathbb{P}^1$. Let g be a generator of $G = Gal(C/\mathbb{P}^1) \cong \mathbb{Z}/3\mathbb{Z}$. Then G acts on the homology group $H_1(C,\mathbb{Z})$ and on the vector space $H^0(C,\omega_C)$ of regular differential forms. We have

$$rank_{\mathbb{Z}} H_1(C,\mathbb{Z}) = 6, \quad \dim_{\mathbb{C}} H^0(C,\omega_C) = 3.$$

Since $\mathbb{P}^1$ has only the trivial regular differential forms, the action of G on $H^0(C,\omega_C)$ must be free (outside 0). Therefore, by Poincaré duality, G acts freely on $H_1(C,\mathbb{Z})$, too (outside 0). The ineffective kernel of the group ring $\mathbb{Z}[G]$ with respect to the action on $H_1(C,\mathbb{Z})$ is the ideal generated by $t := 1+g+g^2$. Namely, $gt = t$ because $g^3 = 1$. Therefore, for each $0 \neq \alpha \in H_1(C,\mathbb{Z})$ it holds that $gt\alpha = t\alpha$, hence $t\alpha = 0$ because of the free action of g.

Now it is clear that the quotient ring $\mathbb{Z}[G]/(1+g+g^2)$ is isomorphic to the ring $\mathfrak{O} = \mathfrak{O}_K$ of integers of the imaginary quadratic number field $K = \mathbb{Q}(\rho), \rho$ a primitive 3-rd unit root. On this way $H_1(C,\mathbb{Z})$

is endowed with the structure of a torsion free $\mathfrak{O}$-module of rank 3. We used the well-known fact that $H_1(C,\mathbb{Z})$ is a torsion free abelian group. Since $\mathfrak{O}$ is a principal domain, $H_1(C,\mathbb{Z})$ is isomorphic to $\mathfrak{O}^3$ as $\mathfrak{O}$-module.

With $\mathbb{Z}[G] \cong \mathbb{Z}[T]/(T^3-1)$ it is easy to check that $\mathbb{Z}[G]^* = \{\pm 1, \pm g, \pm g^2\}$ is the unit group. For this purpose write an element of $\mathbb{Z}[G]$ as $\varepsilon = a + bg + m(1+g+g^2)$, $a, b, m \in \mathbb{Z}$, set

$$E(T) = a + bT + m(1+T+T^2)$$

and look for the existence of a polynomial

$$F(T) = c + dT + n(1+T+T^2), \quad c, d, n \in \mathbb{Z},$$

such that ρ is a zero of $E(T)F(T)-1$. This happens iff $a+b\rho$ is a unit in $\mathfrak{O}^*$ (with inverse $c+d\rho$). Notice that $1+\rho+\rho^2 = 0$.

If $(\alpha_1, \alpha_2, \alpha_3)$ is an $\mathfrak{O}$-basis of $H_1(C,\mathbb{Z})$, then

$$(\alpha_1, \alpha_2, \alpha_3, g_1\alpha_1, g_2\alpha_2, g_3\alpha_3), \quad g_1, g_2, g_3 \in \mathbb{Z}[G]^* \setminus \{\pm 1\}, \tag{1.1}$$

is a $\mathbb{Z}$-basis. Namely, we know that $g_j \in \{\pm g, \pm g^2\}$. Now it is clear that

$$\mathfrak{O}\alpha_j = (\mathbb{Z} + \mathbb{Z}\rho)\alpha_j = \mathbb{Z}\alpha_j + \mathbb{Z}g_j\alpha_j, \quad j = 1, 2, 3,$$

and we get the above $\mathbb{Z}$-basis.

The intersection product of (oriented) cycles on C is denoted by $\circ$. It is skew-symmetric, non-degenerated with values in $\mathbb{Z}$ and unimodular, see $[G-H]$. Therefore there exists a *normal basis* $(\alpha_1, \cdots, \alpha_6)$ of $H_1(C,\mathbb{Z})$ defined by the condition

$$(\alpha_i \circ \alpha_j) = I, \quad I := \begin{pmatrix} O & E_3 \\ -E_3 & O \end{pmatrix}$$

The set of normal bases of $H_1(C,\mathbb{Z})$ is a $\mathbb{S}p(6,\mathbb{Z})$-orbit of (any) one of them, where $\mathbb{S}p(6,\mathbb{Z})$ is the integral symplectic group

$$\mathbb{S}p(6,\mathbb{Z}) = \{S \in \mathbb{G}l_6(\mathbb{Z}); SI^tS = I\}.$$

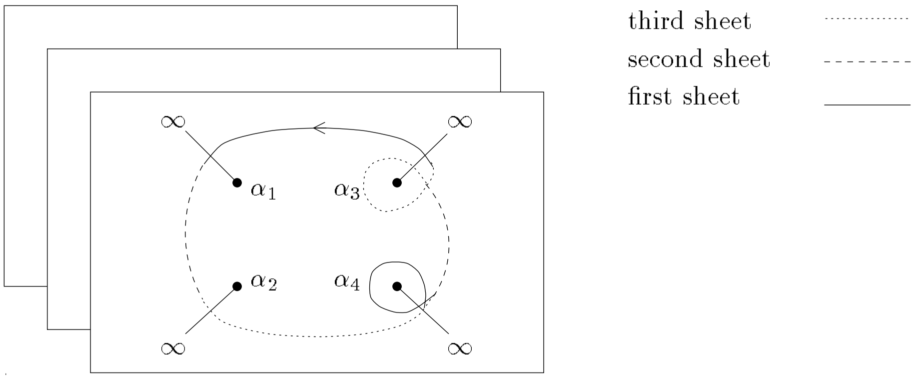
The Galois group action on $H_1(C,\mathbb{Z})$ is compatible with the intersection product, that means $(g\alpha)\circ(g\beta) = \alpha \circ \beta$ for all cycles α, β. In this sense the $\mathbb{Z}[G]-$ and the $\mathfrak{O}$-action is compatible with $\circ$.

Picard observed in [Pic] that there is a nice normal basis of the form (1.1) with $g_1 = -g, g_2 = g, g_3 = g^2$, up to transposition of the third and fourth element. To make it visible he used the classical presentation of C as Riemann surface consisting of three exemplars of $\mathbb{P}^1$ (sheets) connected along 4 cuts joining 4 of the branch points t_1, t_2, t_3, t_4 with the fifths one ∞, say, without loss of generality. The number $k = 5$ of branch points is correct because of the Hurwitz genus formula:

$$-4 = \text{ Euler number of } C = |G| \cdot \text{ Euler number of } \mathbb{P}^1 - k(3-1) = 3 \cdot 2 - 2k.$$

In picture (1.2) we draw as example a non-trivial oriented cycle going through the cuts from one sheet to another.

(1.2)



Definition 1.3 An ordered set of *Picard cycles* is a normal basis of $H_1(C,\mathbb{Z})$ of the form

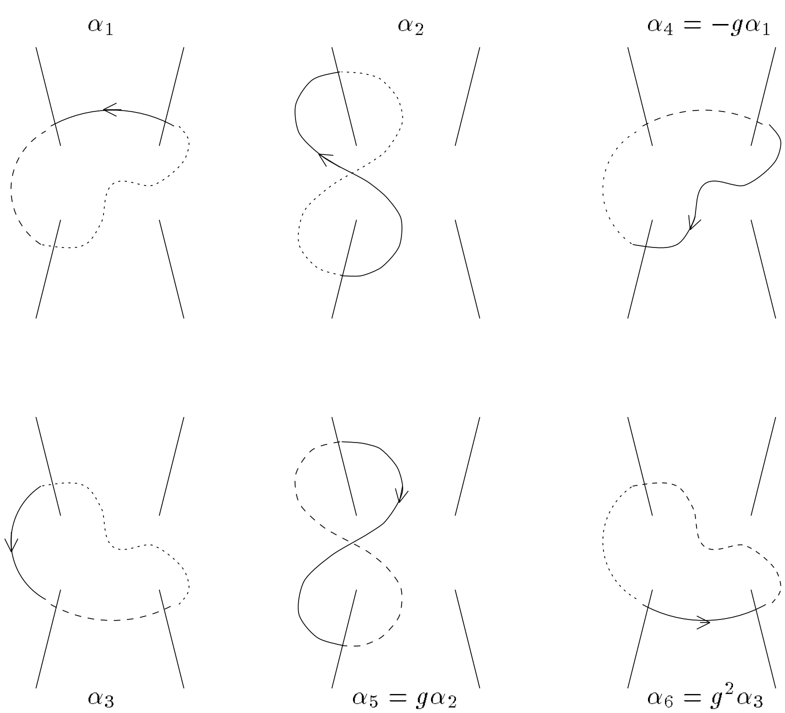
$$*(\alpha_1,\alpha_2,\alpha_3) := (\alpha_1,\alpha_2,-g\alpha_1,\alpha_3,g\alpha_2,g^2\alpha_3).$$

It is also called a *Picard basis* of $H_1(C,\mathbb{Z})$.

Lemma 1.4 *(Picard [Pic]) Picard cycles exist on C.*

For a proof we reproduce in picture (1.5) one set of them in the style of (1.2).

(1.5)



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Let C for a moment be an arbitrary smooth (compact complex algebraic) curve of genus 3. The choices of a normal basis $\alpha = (\alpha_1, \ldots, \alpha_6)$ of $H_1(C,\mathbb{Z})$ and of a $\mathbb{C}$-basis $\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$ of $H^0(C,\omega_C)$, ω_C the sheaf of regular differential forms, define a period matrix

$$\Pi = \Pi(\alpha,\vec{\omega}) = \int_\alpha \vec{\omega} = \left(\int_{\alpha_j} \omega_i\right) \in \ \mathrm{Mat}_{3\times 6}(\mathbb{C}).$$

satisfying Riemann's period relations

$$\Pi I^t\Pi = O, -i\Pi I^t\overline{\Pi} < 0 \tag{1.6}$$

All matrices $\Pi \in \ \mathrm{Mat}_{3\times 6}(\mathbb{C})$ satisfying (1.6) are shortly called period matrices. They appear precisely as period matrices of principally polarized abelian threefolds, see $[G-H]$, II.6. Via base change on $H^0(C,\omega_C)$ each basis α of $H_1(C,\mathbb{Z})$ defines a coset

$$\Pi(\alpha) = \mathbb{G}l_3(\mathbb{C})\Pi(\alpha,\vec{\omega}) \in \mathbb{G}l_3(\mathbb{C}) \setminus \{ \text{ period matrices}\}.$$

The separation of $\Pi = (\Pi_1|\Pi_2)$ of each period matrix into two quadratic matrices defines a map $\Pi \longmapsto \Pi_1^{-1}\Pi_2 \in \mathbb{G}l_3(\mathbb{C})$. Because of the period relations (1.6) the image lies in the hermitian symmetric space (Siegel's upper halfspace)

$$\mathbb{H}_3 = \{\Omega \in \mathbb{G}l_3(\mathbb{C}); \quad {}^t\Omega = \Omega, \quad \mathrm{Im}\ \Omega > 0\}.$$

The above correspondence defines a bijective map

$$\mathbb{G}l_3(\mathbb{C}) \setminus \{ \text{ period matrices } \} \Longleftrightarrow \mathbb{H}_3 \tag{1.7}$$

which gives to the set on the left side a smooth complex structure. We call $\mathbb{H}_3$ shortly the *period space of principally polarized abelian threefolds.* The subset coming from period matrices of smooth curves is denoted by $\mathbb{H}_3^*$. This is an open dense analytic subspace of $\mathbb{H}_3$. We call it the *period space of smooth genus 3 curves.*

The $\mathbb{Z}$-module $\Pi\mathbb{Z}^6$ generated by the colums of Π is the corresponding *period lattice*. The abelian variety $J(C) = \mathbb{C}^3/\Pi\mathbb{Z}^6$ is called the *Jacobian variety* of C. The intersection product on $H_1(C,\mathbb{Z})$ considered as element π of

$$\mathrm{Hom}_{\mathbb{Z}}(\Lambda^2 H_1(C,\mathbb{Z}),\mathbb{Z}) \quad \cong H^2(J(C),\mathbb{Z})$$

is the canonical (principal) polarization of $J(C)$. We call the pair Jac $C = (J(C),\pi)$ shortly the *Jacobian of* C. The following *Torelli theorem* is important:

Theorem 1.8 (Torelli, see e.g. $[G-H]$, II.7). *For smooth curves C, C' it holds that*

$$\mathrm{Jac}\ C \cong \mathrm{Jac}\ C' \text{ iff } \quad C \cong C'.$$

2

This theorem allows to endow the set of isomorphy classes of curves genus 3 with an algebraic structure. First recall from [B-B] that the Baily-Borel compactification $\hat{\mathcal{A}}_3$ of $\mathcal{A}_3 := \mathbb{H}_3/\mathbb{S}p(6,\mathbb{Z})$ is an algebraic variety. $\mathcal{A}_3$ is the moduli space of principally polarized abelian threefolds. It parametrizes precisely the latter objects. A Zariski-open part $\mathcal{A}_3^*$ of $\mathcal{A}_3$ parametrizes precisely the Jacobians of smooth curves of genus 3, hence, by the Torelli theorem, the isomorphy classes of these curves.

Since the multivalence of period matrices of a curve C comes precisely from $\mathbb{C}$-base changes in $H^0(C,\omega_C)$ and $\mathbb{Z}$-base changes of normal bases of $H_1(C,\mathbb{Z})$ realized by integral unimodular symplectic transformations, we can identify $\mathcal{A}_3^*$ with the space of double cosets $\mathbb{G}l_3(\mathbb{C}) \setminus \{ \text{ period matrices } \}^*/\mathbb{S}p(6,\mathbb{Z})$, where the star * indicates that we restrict to period matrices coming from smooth curves. For the relative

Schottky problem it is important to notice that we have the following Zariski-open embeddings

$$\begin{array}{ccc} \mathbb{G}l_3(\mathbb{C}) \setminus \{ \text{ period matrices } \}^*/\mathbb{S}p(6,\mathbb{Z}) & \subset & \mathbb{G}l_3(\mathbb{C}) \setminus \{ \text{ period matrices } \}/\mathbb{S}p(6,\mathbb{Z}) \\ \| & & \| \\ \mathcal{A}_3^* & & \mathcal{A}_3 \end{array} \tag{1.9}$$

2. Period period matrices

We want to prove that our smooth 3-sheeted Galois coverings of genus 3 of $\mathbb{P}^1$ have a nice moduli space sitting in $\mathcal{A}_3^*$ as (Zariski-closed) subvariety. But let us first look for a period space for these curves together with an open analytic embedding into $\mathbb{H}_3^*$. For this purpose we define with a glance to Definition 1.3 the $\mathbb{C}$-linear embedding $* : \mathbb{C}^3 \longrightarrow \mathbb{C}^6$ by

$$\mathfrak{a} = (a_1, a_2, a_3) \longmapsto *\mathfrak{a} := (a_1, a_2, -\rho a_1, a_3, \rho a_2, \rho^2 a_3) \tag{2.1}$$

Set $J = \frac{1}{\sqrt{-3}} I$. It defines a hermitian structure on $\mathbb{C}^6$ of signature (3,3). Its restriction along $*$ yields an hermitian space $(\mathbb{C}^3, <,>)$. An easy calculation shows that its signature is (2,1). More precisely (see [H 95]), it holds that

$$\langle \mathfrak{u}, \mathfrak{v} \rangle = *\mathfrak{u} J^t \overline{*\mathfrak{v}} = \mathfrak{u} \begin{pmatrix} 0 & 0 & \rho \\ 0 & 1 & 0 \\ \overline{\rho} & 0 & 0 \end{pmatrix} {}^t\overline{\mathfrak{v}}. \tag{2.2}$$

Definition 2.3

A period matrix of the form

$$\Pi = \begin{pmatrix} *\mathfrak{a} \\ \overline{*\mathfrak{b}} \\ \overline{*\mathfrak{c}} \end{pmatrix}, \quad \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in V := \mathbb{C}^3 \quad (\text{ row vectors}),$$

is called a *Picard period matrix.*

An easy calculation (see [H 95]) shows that the Riemann period relations (1.6) transfer to the following geometric criterion.

Lemma 2.4 *A matrix* $\Pi \in Mat_{3\times 6}(\mathbb{C})$ *is a Picard period matrix if and only if it has the form*

$$\Pi = \begin{pmatrix} *\mathfrak{a} \\ \overline{*\mathfrak{b}} \\ \overline{*\mathfrak{c}} \end{pmatrix}$$

with the properties

$(-)$ $\qquad \langle \mathfrak{a}, \mathfrak{a} \rangle < 0,$

$(\perp)$ $\qquad \mathfrak{a}^\perp = \mathbb{C}\mathfrak{b} + \mathbb{C}\mathfrak{c},$

where $\perp$ *denotes orthogonality in* V *with respect to the hermitian metric* $<,>$.

2

Proposition 2.5 *Let C be a smooth 3-sheeted Galois covering of $\mathbb{P}^1$ of genus 3. Then C has a Picard period matrix.*

Proof. One establishes a period matrix $\Pi = \Pi(\alpha, \vec{\omega}) = \Pi(*\alpha, \vec{\omega})$ of C by means of Picard cycles $*\alpha$ in the sense of Definition 1.3 and a basis of eigenvectors $\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$ of g operating on $H^0(C, \omega_C)$. Since G acts (outiside of 0) freely on $H^0(C, \omega_C)$, the eigenvalues of g must be different from 1. If η is an eigenform of eigenvalue ρ orρ^2, then $g^*\eta$ is eigenform of eigenvalue ρ^2 or ρ, respectively. Without loss of generality we can assume that the eigensubspace of $H^0(C, \omega_C)$ of eigenvalue ρ is one-dimensional. Otherwise we could change from generator g of G to g^2. Now let $\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$ be an eigenbasis of $H^0(C, \omega_C)$ and $*\alpha$ a Picard basis of $H_1(C, \mathbb{Z})$. It is a matter of linear algebra (see [Pic]) to verify that the period matrix $\Pi(*\alpha, \vec{\omega})$ is of Picard type.

2

We call a complex line L in V *negative*, if it belongs to the negative cone

$$V_- := \{\mathfrak{u} \in V; \langle \mathfrak{u}, \mathfrak{u} \rangle < 0\}$$

By Lemma 2.4 Picard period matrices correspond via $\mathfrak{a} \longmapsto L := \mathbb{C}\mathfrak{a}$ to orthogonal decompositions

$$V = L \bigoplus L^\perp, \quad L \subset V \text{ a negative line .}$$

The linear group $\mathcal{G}l_2 = \mathcal{G}l_2(\mathbb{C})$ acts on $L^\perp$. A basis of $L^\perp$ is uniquely determined up to $\mathcal{G}l_2$-equivalence. Knowing $L^\perp$, the basis vector $\mathfrak{a}$ of L is uniquely determined up to $\mathcal{G}l_1$-equivalence. So from $\mathfrak{a} \in V_-$ one recovers uniquely the corresponding Picard matrix up to $\mathcal{G}l_1 \times \overline{\mathcal{G}l_2}$-equivalence (According to the last two rows of Π in Lemma 2.4 we use the sign of complex conjugation in the second factor). Moreover, we get a bijective correspondence

$$\begin{array}{rcl} \mathcal{G}l_1 \times \overline{\mathcal{G}l_2} \setminus \{\text{Picard matrices}\} & \xrightarrow{\sim} & \mathbb{B} := \mathbb{P}V_- \subset \mathbb{P}V \cong \mathbb{P}^2. \\ \Pi & \longmapsto & \mathbb{P}\mathfrak{a} \end{array} \tag{2.6}$$

The space $\mathbb{B}$ is nothing else but a projective transform of the standard complex unit ball

$$\mathbb{B}_0 = \{(x, y) \in \mathbb{C}^2; \quad |x|^2 + |y|^2 < 1\} \subset \mathbb{C}^2 \subset \mathbb{P}^2.$$

It endows the set on the left side of (2.6) with a complex structure in a natural manner. For such a transform an element of $\mathcal{G}l_3(\mathfrak{O})$ can be used, see (2.2).

With diagonal embedding of $\mathcal{G}l_1 \times \overline{\mathcal{G}l_2}$ into $\mathcal{G}l_3$ and natural identifications we get inclusions of left cosets

$$\begin{array}{rcl} \mathbb{B} = \mathcal{G}l_1 \times \overline{\mathcal{G}l_2} \setminus \{\text{Picard period matrices}\} & \subset & \mathcal{G}l_3 \setminus \{\text{period matrices}\} = \mathbb{H}_3, \\ \mathbb{B}^* := \mathcal{G}l_1 \times \overline{\mathcal{G}l_2} \setminus \{\text{Picard period matrices}\}^* & \subset & \mathcal{G}l_3 \setminus \{\text{period matrices}\}^* = \mathbb{H}_3^*, \end{array} \tag{2.7}$$

where the star * indicates again that we take only period matrices coming from smooth genus 3 curves. Via the correspondence $\Pi = (\Pi_1|\Pi_2) \longmapsto \Pi_1^{-1}\Pi_2$ described in (1.7) the ball $\mathbb{B}$ appears as analytic subvariety in $\mathbb{H}_3$. It can be described by some algebraic equations of degree at most 2 in the coefficients of $\mathbb{H}_3$-matrices. These equations have coefficients in K. Explicitly, this K-quadratic algebraic embedding has been described by Picard in [Pic]. Now we fix a smooth Galois covering C of genus 3 and a Picard basis $*\alpha = *(\alpha_1, \alpha_2, \alpha_3)$ of $H_1(C, \mathbb{Z})$.

2.8 Definitions-Notations. The subgroup of all elements of $\mathbb{S}p(6,\mathbb{Z})$ sending any Picard basis of $H_1(C,\mathbb{Z})$ to another Picard basis is denoted by $\mathbb{S}p(6,\mathbb{Z})_{\mathrm{Pic}}$. The subgroup of $\mathbb{S}p(6,\mathbb{Z})$ generated by all elements sending $*\alpha$ to a Picard basis of $H_1(C,\mathbb{Z})$ is denoted by $\mathbb{S}p(6,\mathbb{Z})_\alpha$. The arithmetic group

$$\mathbb{U}((2,1,),\mathfrak{O}) = \mathbb{U}(<,>,\mathfrak{O}) = \{\gamma \in \mathbb{G}l_3(\mathfrak{O}); \langle \mathfrak{u}\gamma, \mathfrak{v}\gamma\rangle = \langle \mathfrak{u},\mathfrak{v}\rangle \text{ for all } \mathfrak{u},\mathfrak{v} \in \mathbb{C}^3\} \tag{2.9}$$

is called the *Picard modular group* of K with respect to $<,>$.

With the help of (2.2) it is easy to see that $\mathbb{U}(<,>,\mathfrak{O})$ is a $\mathbb{G}l_3(\mathfrak{O})-$conjugate of $\mathbb{U}((2,1),\mathfrak{O})$ which is originally defined by means of the hermitian metric corresponding to the diagonal matrix $\mathrm{diag}(1,1,-1)$ instead of $<,>$. In this sense we use the identifying notation in (2.9). The group $\mathbb{U}((2,1),\mathfrak{O})$ acts on $\mathbb{B}_0$ but $\mathbb{U}(<,>,\mathfrak{O})$ acts on $\mathbb{B}$. The above identification goes conform with an identification of $\mathbb{B}_0$ and $\mathbb{B}$. We identify $H_1(C,\mathbb{Z})$ with $\mathfrak{O}^3$, for example by means of the $\mathfrak{O}$-basis α. Then we get $\mathfrak{O}^3$-representations

$$\mathbb{S}p(6,\mathbb{Z})_{\mathrm{Pic}} \subseteq \mathbb{S}p(6,\mathbb{Z})_\alpha \subseteq \mathbb{G}l_3(\mathfrak{O}).$$

We want to prove now, that these representations are unitary. The proof of Proposition 2.5 shows that

$$\Pi = \Pi(*\alpha,\vec{\omega}) = \begin{pmatrix} *\mathfrak{a} \\ \overline{*\mathfrak{b}} \\ \overline{*\mathfrak{c}} \end{pmatrix},$$

$\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$ an eigenbasis of g of eigenvalues ρ,ρ^2 in this order, is a Picard matrix. If $\gamma \in \mathbb{S}p(6,\mathbb{Z})_\alpha \subseteq \mathbb{G}l_3(\mathfrak{O})$ is one of the generators sending $*\alpha$ to the Picard basis $*(\alpha\gamma)$, then also

$$\Pi(*(\alpha\gamma),\vec{\omega}) = \begin{pmatrix} *(\mathfrak{a}\gamma) \\ \overline{*(\mathfrak{b}\gamma)} \\ \overline{*(\mathfrak{c}\gamma)} \end{pmatrix}$$

is a Picard period matrix. Since both $*\alpha$ and $*(\alpha\gamma)$ are normal bases, we can find a symplectic matrix $\Sigma \in \mathbb{S}p(6,\mathbb{Z})$ such that $*(\alpha\gamma) = (*\alpha)\Sigma$, hence

$$\Pi(*(\alpha\gamma),\vec{\omega}) = \Pi(*\alpha,\vec{\omega})\Sigma$$

or

$$\begin{pmatrix} *(\mathfrak{a}\gamma) \\ \overline{*(\mathfrak{b}\gamma)} \\ \overline{*(\mathfrak{c}\gamma)} \end{pmatrix} = \begin{pmatrix} *\mathfrak{a} \\ \overline{*\mathfrak{b}} \\ \overline{*\mathfrak{c}} \end{pmatrix}\Sigma$$

Therefore for each vector $\mathfrak{u} \in \{\mathfrak{a},\mathfrak{b},\mathfrak{c}\} \subset \mathbb{C}^3$ it holds that $*(\mathfrak{u}\gamma) = (*\mathfrak{u})\Sigma$, and for each pair $\mathfrak{u},\mathfrak{v}$ of the same set we get

$$\langle \mathfrak{u}\gamma,\mathfrak{v}\gamma\rangle = *(\mathfrak{u}\gamma)J^t\overline{*(\mathfrak{v}\gamma)} = (*\mathfrak{u})\Sigma J^t\Sigma^t\overline{*\mathfrak{v}} = (*\mathfrak{u})J^t\overline{*\mathfrak{v}} = \langle \mathfrak{u},\mathfrak{v}\rangle$$

because Σ is symplectic, hence $\Sigma J^t\Sigma = J$. This means that γ belongs to $\mathbb{U}((2,1),\mathfrak{O})$. Altogether we have inclusions

$$\mathbb{S}p(6,\mathbb{Z})_{\mathrm{Pic}} \subseteq \mathbb{S}p(6,\mathbb{Z})_\alpha \subseteq \mathbb{U}((2,1),\mathfrak{O}) \subset \mathbb{G}l_3(\mathfrak{O}). \tag{2.10}$$

3. Monodromy groups

Now consider the algebraic curve family $\mathfrak{C}/T \subset \mathbb{P}^2 \times T/T$, $\quad T$ the 4-dimensional affine space, $T = T(\mathbb{C}) = \mathbb{C}^4$, which fibres are the projective closures C_t in $\mathbb{P}^2$ of the affine plane curves defined by the equations $Y^3 = (X-t_1)(X-t_2)(X-t_3)(X-t_4)$ for each $t = (t_1, t_2, t_3, t_4) \in T$. The fibre curve C_t is smooth if and only if t does not belong to one of the six hyperplanes $H_{jk} : t_j = t_k, 1 \leq j < k \leq 4$, of T. Let T^* denote the complement of these hyperplanes in T. The restricted subfamily $\mathfrak{C}^*/T^*$ over T^* is smooth. Each fibre is a smooth Galois covering of $\mathbb{P}^1$ of genus 3. The Galois action comes from $(x, y) \longmapsto (x, \rho y)$. For the genus one has to apply the genus formula

$$\text{genus } (D) = (d-1)(d-2)/2$$

for smooth curves $D \subset \mathbb{P}^2$ of degree d to our quartics C_t with homogeneous equations

$$F_t(W, X, Y) = WY^3 - (X - t_1W)(X - t_2W)(X - t_3W)(X - t_4W). \tag{3.1}$$

The space T^* is the configuration space of braids with 4 strings. For definitions and elementary properties we refer to [Han]. The symmetric group S_4 of four elements acts on T^* by permution of coordinates. It defines an unramified Galois covering $T^* \longrightarrow T^*/S_4$, hence an exact sequence with fundamental groups

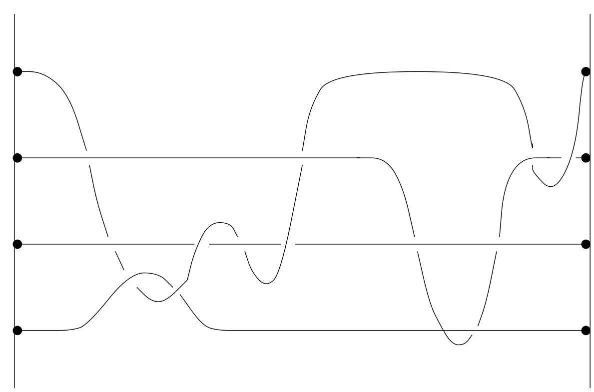
$$1 \longrightarrow \pi_1(T^*) \longrightarrow \pi_1(T^*/S_4) \longrightarrow S_4 \longrightarrow 1. \tag{3.2}$$

The fundamental group of T^*/S_4 is isomorphic to the *(Fox) braid group* of (four) strings. We keep in mind number 4 of strings and use the notation

$$\mathfrak{Z} = \pi_1(T^*/S_4) \cong \pi_1(T^*/S_4, P_0), P_0 \in T^*/S_4.$$

The *coloured braid group* $\mathfrak{Z}_{\text{col}}$, *or (Artin) group of coloured strings*, consists of braids with identical permutation of starting and end points of strings. For a good imagination a typical element of $\mathfrak{Z}_{\text{col}}$ is drawn in picture (3.3).

(3.3)



The coloured braid group is isomorphic to the fundamental group of the configuration space T^*.

$$\mathfrak{Z}_{\text{col}} = \pi_1(T^*) \cong \pi_1(T^*, \tau), \tau \in T^*.$$

So we have an obvious version

$$1 \longrightarrow \mathfrak{Z}_{\text{col}} \longrightarrow \mathfrak{Z} \longrightarrow S_4 \longrightarrow 1 \tag{3.4}$$

of the exact sequence of abstract fundamental groups (3.2), which is called the *braid group sequence.*

The action of braid groups on the homology group $H_1(C_t,\mathbb{Z})$ of a curve C_t which belongs to the smooth curve family $\mathfrak{C}^*$ over T^* is nicely explained in [Arn]. We restrict ourselves to the curve family (3.1). The authors of [Arn] substitute the big configuration space $T^* = F_4(\mathbb{C})$ by the smaller one $F_4(\mathbb{D}) = T^* \cap \mathbb{D}^4$, where $\mathbb{D}$ denotes a subdisc of $\mathbb{C}$ around 0, say, but this change is not necessary for us because of homotopy equivalence of the spaces. All the homology groups $H_1(C_t,\mathbb{R})$ or $H_1(C_t,\mathbb{C})$ can be considered as fibres of a (locally trivial) fibration $\mathfrak{H}_1(\mathfrak{C}^*,\mathbb{R})$ or $\mathfrak{H}_1(\mathfrak{C}^*,\mathbb{C})$ over T^* with the embedded local system $\mathfrak{H}_1(\mathfrak{C}^*,\mathbb{Z})$. Each path w in T^* joining the points t and s, say, induces an isomorphism

$$T_w : H_1(C_t,\mathbb{Z}) \xrightarrow{\sim} H_1(C_s,\mathbb{Z}). \tag{3.5}$$

If s and t have the same image in T^*/S_4, that means s and t have the same coordinates up to a permutation, then the curves C_s and C_t coincide. The path w goes down to a cycle on T^*/S_4. On this way we get an action of the fundamental group $\mathfrak{Z} = \pi_1(T^*/S_4)$ on $H_1(C_t,\mathbb{Z})$. The coordinates t_j of t are understood as the four finite branch points of the 3-sheeted Galois covering $C_t \longrightarrow \mathbb{P}^1$. We refer now to picture (1.2) in order to make this action more visible. Going contineously along a cycle $w : [0,1] \longrightarrow T^*$ of T^* starting and ending at t we move the branch points. This movement is understood as a deformation of C_t coming back to the same curve at the end. Now fix a cycle α representing an element of the homology group $H_1(C_t,\mathbb{Z})$. Restricting w to $[0,r]$ for $0 \le r \le 1$ we dispose on partial pathes w_r of w joining t and $w(r)$. Denote by T_r the isomorphism (3.5) with $w = w_r$. The cycles $T_r(\alpha)$ are understood as deformations of the cycle α, and $T_1(\alpha)$ coincides with $T_w(\alpha)$. Altogether one obtains a rather obvious imagination of the action of $\mathfrak{Z}_{\text{col}} = \pi_1(T^*)$ on $H_1(C_t,\mathbb{Z})$ originally used already by Picard [Pic] and Alezais [Ale]. They applied it to the most simple generators of $\mathfrak{Z}_{\text{col}}$ represented by moving a branch point around another along a circle. Then they observe the corresponding deformations of the original Picard cycles (1.5). The same can be done with the most natural generators of $\mathfrak{Z}$ moving along pathes which transpose two branch points. On this way one gets explicit representations

$$\mathfrak{Z}_{\text{col}} = \pi_1(T^*) \longrightarrow \text{ Aut } H_1(C_t,\mathbb{Z}), \qquad \mathfrak{Z} = \pi_1(T^*/S_4) \longrightarrow \text{ Aut } H_1(C_t,\mathbb{Z}). \tag{3.6}$$

It is quite clear that the isomorphisms (3.5) preserve the intersection product of oriented cycles. Therefore we dispose on natural symplectic representations. Moreover, also relations $\beta = \pm g\alpha, \quad g$ a generator of the Galois group of $C_t \longrightarrow \mathbb{P}^1, \quad \alpha,\beta \in H_1(C_t,\mathbb{Z})$, are preserved. With the notations of (2.10) we get group homomorphisms

$$\mathfrak{Z}_{\text{col}} = \pi_1(T^*) \subset \mathfrak{Z} = \pi_1(T^*/S_4) \longrightarrow \mathbb{S}p(6,\mathbb{Z})_{\text{Pic}} \subseteq \mathbb{S}p(6,\mathbb{Z})_\alpha \subseteq \mathbb{U}((2,1),\mathfrak{O}) \tag{3.7}$$

Definition 3.8 The image of $\mathfrak{Z} = \pi_1(T^*/S_4)$ in $\mathbb{U}((2,1),\mathfrak{O})$ is called the *monodromy group of the family* $\mathfrak{C}^*/T^*$. The corresponding images of supgroups U of $\mathfrak{Z}$ are denoted by Mon U.

Next we will restrict the curve family to the subspace of T_0^* defined by

$$T_0 := \{t = (t_1,t_2,t_3,t_4) \in T; \quad tr(t) := t_1+t_2+t_3+t_4 = 0\} \quad , \quad T_0^* := T^* \cap T_0.$$

The Tschirnhaus map $p = id - \frac{1}{4}\delta \circ tr, \quad \delta : \mathbb{C} \longrightarrow T = \mathbb{C}^4$ the diagonal embedding, projects T onto T_0 and T^* onto T_0^*. The fibres of p are isomorphic to $\mathbb{C}$. For fibre spaces $E \longrightarrow X$ with fibres isomorphic to F one has a long exact sequence of homotopy groups

$$\ldots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(X) \longrightarrow \pi_{n-1}(F) \longrightarrow \ldots \tag{3.9}$$

(see e.g. [Hu]). Especially for $n = 1$ we get the part

$$\pi_1(F) \longrightarrow \pi_1(E) \longrightarrow \pi_1(X) \longrightarrow \pi_0(F). \tag{3.10}$$

The application to $T^* \longrightarrow T_0^*$ yields an isomorphism of $\pi_1(T^*)$ and $\pi_1(T_0^*)$ because the fibres $\mathbb{C}$ are connected and simply-connected, hence $\pi_0(\mathbb{C})$ and $\pi_1(\mathbb{C})$ are trivial. In the same manner one gets an isomorphism $\pi_1(T^*/S_4) \cong \pi_1(T_0^*/S_4)$ because the fibres of $T^*/S_4 \longrightarrow T_0^*/S_4$ coincide also with $\mathbb{C}$. The sequence part of (3.9) for $n = 2$ applied to $T^* \longrightarrow T_0^*$ yields $\pi_2(T^*) \cong \pi_2(T_0^*)$. Since $\pi_2(T^*)$ is trivial, see [Han], page 13, we get also $\pi_2(T_0^*) = 1$. The Galois covering $T_0^* \longrightarrow T_0^*/S_4$ can be considered as a fibration with discrete fibres consisting of 24 different points identified with the elements of S_4. From the corresponding exact sequence (3.9) and the knowledge of $\pi_0(S_4) = S_4, \pi_j(S_4) = 0$ for $j > 0$, one gets also $\pi_2(T_0^*/S_4) = 1$.

If we restrict $\mathfrak{C}$ to T_0^*, then no isomorphy class of curves of type (3.1) is lost because the restricted family $\mathfrak{C}_0^*/T_0^*$ consists precisely of all Tschirnhaus transforms of members of the starting family $\mathfrak{C}^*/T^*$. Without loss of information we can change the family to get the same representations of fundamental group $\pi_1(T_0^*) = \mathfrak{Z}_{\text{col}}$ and also of $\mathfrak{Z}$ on each fibre homolgy group $H_1(C_t, \mathbb{Z}), t \in T_0^*$. In correspondence with Definition (3.8) the image of $\mathfrak{Z}$ in $\mathbb{U}((2,1), \mathfrak{O})$ is also the monodromy group of the family $\mathfrak{C}_0^*/T_0^*$. More precisely, for fixed $t \in T_0^*$ the homomorphisms (3.6) change to

$$\mathfrak{Z}_{\text{col}} = \pi_1(T_0^*) \longrightarrow \text{ Aut } H_1(C_t, \mathbb{Z}) \quad , \quad \mathfrak{Z} = \pi_1(T_0^*/S_4) \longrightarrow \text{ Aut } H_1(C_t, \mathbb{Z}), \tag{3.11}$$

and the representations (3.7) are the same as

$$\mathfrak{Z}_{\text{col}} = \pi_1(T_0^*) \subset \mathfrak{Z} = \pi_1(T_0^*/S_4) \longrightarrow \mathbb{S}p(6, \mathbb{Z})_{\text{Pic}} \subseteq \mathbb{S}p(6, \mathbb{Z})_\alpha \subseteq \mathbb{U}((2,1), \mathfrak{O}). \tag{3.12}$$

The projective space $\mathbb{P}T_0 = (T_0 \setminus \{0\})/\mathbb{C}^*$ is identified with the projective plane $\mathbb{P}^2$. The image of T_0^* along the quotient map $T_0 \setminus \{0\} \longrightarrow \mathbb{P}^2$ is shortly denoted by $\mathbb{P}_2^*$. The complement $\mathbb{P}^2 \setminus \mathbb{P}_2^*$ consists of six projective lines $L_{ij}, 1 \leq i < j \leq 4$ going through pairs $P_i, P_j \in \mathbb{P}^2$ of a quadruple $\{P_1, P_2, P_3, P_4\}$ of four points on $\mathbb{P}^2$ in general position:

$$\mathbb{P}_2^* = \mathbb{P}^2 \setminus \Delta, \quad \Delta = \bigcup_{1 \leq i < j \leq 4} L_{ij}$$

The fibration $T_0^* \longrightarrow \mathbb{P}_2^*$ with fibres isomorphic to $\mathbb{C}^*$ yields

$$\begin{gathered} \pi_2(T_0^*) = 1 \longrightarrow \pi_2(\mathbb{P}_2^*) \longrightarrow \\ \mathbb{Z} \cong \pi_1(\mathbb{C}^*) \longrightarrow \mathfrak{Z}_{\text{col}} = \pi_1(T_0^*) \longrightarrow \pi_1(\mathbb{P}_2^*) \longrightarrow 1 = \pi_0(\mathbb{C}^*) \end{gathered} \tag{3.13}$$

as part of the corresponding long exact homotopy sequence (3.9). It is not difficult to see that the image of $\pi_1(\mathbb{C}^*)$ in $\pi_1(T_0^*)$ appears as G-action on $H_1(C_t, \mathbb{Z})$. Namely, the fibre over $\mathbb{P}t = (t_1 : t_2 : t_3 : t_4) \in \mathbb{P}_2^*$ is $\mathbb{C}^* t$. On $\mathbb{C}^* t$ all curves of our family $\mathfrak{C}_0^*/T_0^*$ are isomorphic. Explicitly, after X-coordinate changes, it is easy to see that the restricted curve family along $\mathbb{C}^* t$ is isomorphic to $\{C_\mu : \mu Y^3 = \prod_{i=1}^{4} (X - t_i); \quad \mu \in \mathbb{C}^*\}$ which members are obviously isomorphic.

Moreover, a the simple loop $\mu(r) = e^{2\pi i r}, 0 \leq r \leq 1$, on $\mathbb{C}^*$ around 0 turns any cycle $\alpha = \{(x(\lambda), y(\lambda)); 0 \leq \lambda \leq 1\}$, on $C_1 \cong C_t$ to the cycle $g\alpha = \{(x(\lambda), \rho y(\lambda))\}$ because $\mu\left(\frac{1}{3}\right) = \rho$ and the isomorphisms $C_1 \cong C_\mu : (\mu^{\frac{1}{3}} Y)^3 = \Pi_{i=1\ldots 4}(X - t_i)$ are realised by the Y-coordinate changes $Y' = \mu^{\frac{1}{3}} Y$.

Identifying the image of $\pi_1(\mathbb{C}^*)$ in $\mathfrak{Z}_{\text{col}}$ with the Galois group $G \cong \mathbb{Z}/3\mathbb{Z}$ we see that

$$G \cong \text{ Mon } G = \langle \rho E_3 \rangle \subset \Gamma := \mathbb{U}((2,1), \mathfrak{O}). \tag{3.14}$$

The monodromy subgroup of $G \subset \mathfrak{Z}_{\mathrm{col}} \subset \mathfrak{Z}$ appears as center of Γ. Knowing the image $G \cong \mathbb{Z}/3\mathbb{Z}$ of the third homomorphism in (3.13) we can single out the short exact sequencc

$$1 \longrightarrow G \longrightarrow \mathfrak{Z}_{\mathrm{col}} \longrightarrow \pi_1(\mathbb{P}_2^*) \longrightarrow 1 \tag{3.15}$$

of group homomorphisms.

The S_4-action on T_0^* goes down to $\mathbb{P}T_0^* = \mathbb{P}_2^*$ such that the $\mathbb{C}^*$-fibration $T_0^* \longrightarrow \mathbb{P}_2^*$ is S_4-equivariant. The quotient fibration $T_0^*/S_4 \longrightarrow \mathbb{P}_2^*/S_4$ has everywhere the fibres $\mathbb{C}^*$, too. In the same manner as above for (3.13) we get the exact homotopy sequence

$$\begin{aligned} &\pi_2(T_0^*/S_4) = 1 \longrightarrow \pi_2(\mathbb{P}_2^*/S_4) \longrightarrow \mathbb{Z} = \pi_1(\mathbb{C}^*) \longrightarrow \\ &\longrightarrow \mathfrak{Z} = \pi_1(T_0^*/S_4) \longrightarrow \pi_1(\mathbb{P}_2^*/S_4) \longrightarrow 1 = \pi_0(\mathbb{C}^*). \end{aligned} \tag{3.16}$$

The commutative diagram

$$\begin{array}{ccc} T_0^* & \rightarrow & \mathbb{P}_2^* \\ \downarrow & & \downarrow \\ T_0^*/S_4 & \rightarrow & \mathbb{P}_2^*/S_4 \end{array}$$

yields a morphism from the homotopy sequence (3.13) to the homotopy sequence (3.16). The corresponding group homomorphisms $\pi_2(\mathbb{P}_2^*) \to \pi_2(\mathbb{P}_2^*/S_4)$ and $\pi_1(\mathbb{C}^*) \to \pi_1(\mathbb{C}^*)$ are isomorphisms. This leads to the exact right part

$$\begin{array}{ccccccccc} 1 & \rightarrow & G & \rightarrow & \mathfrak{Z}_{\mathrm{col}} & \rightarrow & \pi_1(\mathbb{P}_2^*) & \rightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & G & \rightarrow & \mathfrak{Z} & \rightarrow & \pi_1(\mathbb{P}_2^*/S_4) & \rightarrow & 1 \end{array} \tag{3.17}$$

The vertical arrows correspond to embeddings of normal subgroups.

Knowing $\mathfrak{Z}/\mathfrak{Z}_{\mathrm{col}} \cong S_4$ we get the exact diagram

$$\begin{tikzcd} & & G \arrow[dl] \arrow[d] & & \\ 1 \arrow[r] & \mathfrak{Z}_{\mathrm{col}} \arrow[r] \arrow[d] & \mathfrak{Z} \arrow[r] \arrow[d] & S_4 \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \pi_1(\mathbb{P}_2^*) \arrow[r] & \pi_1(\mathbb{P}_2^*/S_4) \arrow[r] & S_4 \arrow[r] & 1 \end{tikzcd} \tag{3.18}$$

with twice the kernel $G \cong \mathbb{Z}/3\mathbb{Z}$.

Now consider the principal ideal $(\sqrt{-3})$ of $\mathfrak{O}$ generated by $\sqrt{-3}$ (or $\varrho - \varrho^2$). Because of the prime decomposition $(3) = (\sqrt{-3})^2$ in $\mathfrak{O}$ it holds that $\mathfrak{O}/(\sqrt{-3})$ is the Galois field $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$. The congruence subgroup $\Gamma(\sqrt{-3})$ of the prime ideal $\sqrt{-3}$ is defined as kernel of the reduction homomorphism

$$\Gamma = \mathbb{U}((2,1),\mathfrak{O}) \longrightarrow \mathbb{U}((2,1),\mathfrak{O}/(\sqrt{-3}) = \mathbb{U}((2,1),\mathbb{F}_3).$$

The latter group is isomorphic to S_4. So we despose on an exact sequence

$$1 \to \Gamma(\sqrt{-3}) \to \Gamma \to S_4 \to 1. \tag{3.19}$$

Proposition 3.20 *With the above notations we have a commutative diagram of group homomorphisms*

$$
\begin{array}{cccccccc}
 & \mathfrak{Z} & \twoheadrightarrow & \mathrm{Mon}\mathfrak{Z} & \subset \mathbb{S}p(6,\mathbb{Z})_{\mathrm{Pic}} \subset \mathbb{S}p(6,\mathbb{Z})_{\alpha} & \subset & & \Gamma \\
(S_4) & \uparrow & & \uparrow & & & (S_4) & \uparrow \\
 & \mathfrak{Z}_{\mathrm{col}} & \twoheadrightarrow & \mathrm{Mon}\mathfrak{Z}_{\mathrm{col}} & \hookrightarrow & & & \Gamma(\sqrt{-3}) \\
 & \uparrow & & \uparrow & & & & \uparrow \\
 & G & \xrightarrow{\sim} & \mathrm{Mon}\, G & \xrightarrow{=} & & & \langle \varrho E \rangle
\end{array} \qquad (3.21)
$$

with vertical inclusions.

Proof. The upper row comes from (3.12) and the lower one from (3.14). The vertical embeddings are clear. The notation (S_4) means that the corresponding factor groups are isomorphic to S_4. It remains to prove that the monodromy group of $\mathfrak{Z}_{\mathrm{col}}$ sits in $\Gamma(\sqrt{-3})$. This has been checked already by Picard [Pic].

In our actual language he worked in the following manner: It is clear that it is only necessary to check that the monodromy representation of a suitable finite system of generators of $\mathfrak{Z}_{\mathrm{col}}$ are $\Gamma(\sqrt{-3})$-matrices. Not knowing braid groups, but with a good feeling, he picked out generators of $\pi_1(\mathbb{P}_2^*)$ in a natural manner. Pulling them back to $\mathfrak{Z}_{\mathrm{col}}$ according to the first row of (3.17) one gets together with G finitely many generators of $\mathfrak{Z}_{\mathrm{col}}$. From the action of the generators on a special Picard basis α of $H_1(C_t,\mathbb{Z}) \simeq \mathfrak{O}^3$ he concluded that the monodromy representants of the generators belong to $\Gamma(\sqrt{-3})$.

2

4. Projective Monodromy

The aim of our next steps is to prove that the right column of (3.21) coincides with the middle column. Then the embeddings of the first row have to be equalities. We need two difficult results, the first one due to Mostow-Deligne, the second one comes from fine surface classification due to the author. For a clear understanding of Mostow-Deligne's result we must distinguish between monodromy and projective monodromy. We change to the projective arithmetic group $\mathbb{P}\Gamma = \mathbb{P}\mathbb{U}((2,1),\mathfrak{O})$ defined as image of Γ in the projective group $\mathbb{P}\mathbb{G}l_3(\mathbb{C})$ along the natural projection $\mathbb{G}l_3(\mathbb{C}) \to \mathbb{P}\mathbb{G}l_3(\mathbb{C})$. More generally we denote by $\mathbb{P}U$ the image of any subgroup U of Γ along the same map. The intersection of Γ with the center of $\mathbb{G}l_3(\mathbb{C})$ is $\langle \varrho E \rangle$. Applying the projectivisation to diagram (3.21) one obtains a commutative diagram

$$
\begin{array}{ccccccccc}
 & \mathfrak{Z} & \twoheadrightarrow & \mathbb{P}\mathrm{Mon}\mathfrak{Z} & \cong & \pi_1(\mathbb{P}_2^*/S_4) & \hookrightarrow & \mathbb{P}\Gamma & \\
(S_4) & \uparrow & & \uparrow & & & & \uparrow & (S_4) \\
 & \mathfrak{Z}_{\mathrm{col}} & \twoheadrightarrow & \mathbb{P}\mathrm{Mon}\mathfrak{Z}_{\mathrm{col}} & \cong & \pi_1(\mathbb{P}_2^*) & \hookrightarrow & \mathbb{P}\Gamma(\sqrt{-3}), &
\end{array} \qquad (4.1)
$$

with vertical embeddings, see (3.17), (3.18) for the isomorphies.

4.2 Definition: The subgroup $\mathbb{P}\mathrm{Mon}\mathfrak{Z}$ ($\mathbb{P}\mathrm{Mon}\mathfrak{Z}_{\mathrm{col}}$) of $\mathbb{P}\Gamma$ is called the (coloured) *projective monodromy group* of the curve families $\mathfrak{C}_0^*/T_0^*$ and $\mathfrak{C}^*/T^*$.

On this precise way we obtained faithful projective representations of $\pi_1(\mathbb{P}_2^*/S_4)$ and $\pi_1(\mathbb{P}_2^*)$ in $\mathbb{P}\mathbb{G}l_3(\mathfrak{O})$. Picard and Mostow-Deligne work with a subfamily $\mathfrak{C}_{01}^*/\mathbb{A}_2^*$ of $\mathfrak{C}^*/T^*$ containing up to isomorphy all smooth Picard curves as fibres. We define

$$
\begin{array}{rcl}
\mathbb{A}_2^* & = & \{(t_1,t_2) \in \mathbb{C}^2; 0,1 \neq t_1 \neq t_2 \neq 0,1\} \subset \mathbb{A}^2(\mathbb{C}) = \mathbb{C}^2, \\
C_{t'} & : & Y^3 = X(X-1)(X-t_1)(X-t_2) \quad \text{(projective closure in } \mathbb{P}^2\text{)},
\end{array}
$$

for $t' = (t_1, t_2) \in \mathbb{A}_2^*$. Observe that we can and will identify

$$\mathbb{A}_2^* = \mathbb{A}^2 \setminus \{5 \text{ lines }\} = \mathbb{P}^2 \setminus \{6 \text{ lines }\} = \mathbb{P}_2^*.$$

By variation of t' we get the Picard family

$$\mathfrak{C}_{01}^*/\mathbb{A}_2^* = \mathfrak{C}_{01}^*/\mathbb{P}_2^* \subset \mathfrak{C}_0^*/T_0^*.$$

By Tschirnhaus transformation $p = \text{ id } - \frac{1}{4}\delta \circ tr$ we get an isomorphism with a subfamily $\mathfrak{C}'^*/\mathbb{P}_2^*$ of $\mathfrak{C}_0^*/T_0^*$. The basic families are up to isomorphy connected by base changes:

$$\begin{array}{rcll} & \mathfrak{C}^* & \rightarrow & T \\ & \downarrow & & \downarrow p \\ & \mathfrak{C}_0^* & \rightarrow & T_0^* \\ & \downarrow & & \downarrow \mathbb{P} \\ \mathfrak{C}_{01}^* \simeq & \mathfrak{C}'^* & \rightarrow & \mathbb{P}_2^* \simeq \mathbb{A}_2^* \simeq \mathbb{P}T_0^*. \end{array}$$

The family $\mathfrak{C}'^*/\mathbb{P}_2^*$ is indeed the most naturel and thinnest one of smooth Picard curves, including all of them up to isomorphy, we can find. The basic space $\mathbb{P}_2^*$ is a small Galois cover of the moduli space of Picard curves:

4.3 Theorem ([H86]). *The quotient surface $\mathbb{P}_2^*/S_4$ is the moduli space of smooth Picard curves.*

2

More precisely, the correspondence $C_{t'} \mapsto \mathbb{P}p(t'), t' = (t_1, t_2)$ as above, yields a bijection between the isomorphy classes of Picard curves and $\mathbb{P}_2^*/S_4$.

The above diagram induces a base change diagram of fibrations

$$\begin{array}{ccc} \mathcal{H}_1(\mathfrak{C}^*/T^*) & \rightarrow & T^* \\ \downarrow & & \downarrow \\ \mathcal{H}_1(\mathfrak{C}_0^*/T_0^*) & \rightarrow & T_0^* \\ \downarrow & & \downarrow \\ \mathcal{H}_1(\mathfrak{C}^*/\mathbb{P}_2^*) & \rightarrow & \mathbb{P}_2^* \end{array}$$

along which we identify the projective representations of fundamental groups of the base spaces in the homology groups $H_1(C_t, \mathbb{Z}) \cong \mathfrak{O}^3$.

Let us recapitulate our kinds of representations. Fix a Picard curve $C = C_t, t \in T^*$. Let us call the natural representation of $\mathfrak{Z}$ in $H_1(C_t, \mathbb{Z})$ the $\mathbb{Z}$-representations of $\mathfrak{Z}$ at t. With a fixed $\mathbb{Z}$-basis one gets a group homomorphism $\mathfrak{Z} \rightarrow \mathbb{G}l_6(\mathbb{Z})$. Taking the $\mathfrak{O}$-structure of $H_1(C_t, \mathbb{Z})$ in our consideration one gets an $\mathfrak{O}^3$-representation. For a fixed $\mathfrak{O}$-Basis α it corresponds to a group homomorphism $\mathfrak{Z} \rightarrow \mathbb{G}l_3(\mathfrak{O})$. Applying $\int \vec{\omega}$ with fixed basis $\vec{\omega}$ of $H^0(C, \omega_C)$ to the cycles the representation changes to a dual $\mathbb{C}^3$-representation. Now choose $\vec{\omega}$ as eigenbasis of $H^0(C, \omega_C)$ and $*\alpha$ a Picard basis of $H_1(C, \omega_C)$. We will take

$$\omega_1 = dx/y, \quad \omega_2 = dx/y^2, \omega_3 = xdx/y^2$$

simultaneously for all $t \in T^*$. Remember to $V = (\mathbb{C}^3, \langle, \rangle)$ and the Picard period matrix $\Pi(*\alpha, \vec{\omega})$, see Lemma 2.4. Looking at the first row we get a V-representation of $\mathfrak{Z}$, more precisely, via $\mathfrak{Z}$-action on Picard bases, a group homomorphism $\mathfrak{Z} \rightarrow \Gamma = \mathbb{U}((2,1), \mathfrak{O})$, which goes down to a projective representation

$$\mathbb{P}\mathfrak{Z} := \mathfrak{Z}/G \cong \pi_1(\mathbb{P}_2^*/S_4) \rightarrow \mathbb{P}\Gamma = \mathbb{P}\mathbb{U}((2,1), \mathfrak{O}).$$

The Picard cycle $*\alpha_0 = *\alpha(\tilde{t}_0), \tilde{t}_0 \in T^*$ (instead of t) fixes a ball point

$$b_0 = b_\alpha(\tilde{t}_0) = (\int_{\alpha_1} \vec{\omega} : \int_{\alpha_2} \vec{\omega} : \int_{\alpha_3} \vec{\omega}) \in \mathbb{B} = \mathbb{P}V_-.$$

The action of $\mathbb{P}\mathfrak{Z}$ and Γ yields a suborbit

$$\mathrm{Mon}(\pi_1(\mathbb{P}_2^*/S_4))b_0 \subseteq \Gamma b_0 = (\mathbb{P}\Gamma)b_0 \subseteq \mathbb{B}$$

of the Γ-orbit Γb_0 of b_0 in $\mathbb{B}$. The same is true for $\mathfrak{Z}_{\mathrm{col}}$, $\mathrm{Mon}\pi_1(\mathbb{P}_2^*)$ and $\Gamma(\sqrt{-3})$, that means we get a suborbit

$$\mathrm{Mon}\pi_1(\mathbb{P}_2^*)b_0 \subseteq \Gamma(\sqrt{-3})b_0 \subset \mathbb{B}$$

If we move $\tilde{t}$ in T^* and the Picard cycle $\alpha(\tilde{t})$ contineously starting from $\tilde{t}_0$ and Picard cycle α_0, say, then we move the suborbits. On a simply-connected open analytic neighbourhood $\tilde{U}$ of $\tilde{t}_0$ we get unique suborbits Mon $\pi_1(\mathbb{P}_2^*/S_4)b(\tilde{t}) \subset \mathbb{B}, \tilde{t} \in \tilde{U}$. Instead of $\tilde{U} \subset T^*$ we can restrict to $\tilde{U} \subset T_0^*$ and moreover to simply-connected neighbourhoods $U = \mathbb{P}\tilde{U}$ of $t_0 = \mathbb{P}\tilde{t}_0 \in \mathbb{P}_2^*$ not changing the orbits.

4.4 Theorem. (Mostow-Deligne [M-D]).

1. $\mathrm{Mon}\pi_1(\mathbb{P}_2^*)$ *is a lattice in* $\mathbb{P}\mathbb{U}((2,1),\mathfrak{O})$, *this means that* $\mathrm{Mon}\pi_1(\mathbb{P}_2^*)$ *has finite index in* $\mathbb{P}\Gamma(\sqrt{-3})$ *or:* $\mathrm{Mon}\pi_1(\mathbb{P}_2^*)$ *is a subgroup of finite index of* $\mathbb{P}\Gamma(\sqrt{-3})$.
2. For a suitable simply-connected open analytic subset U of $\mathbb{P}_2^*$ the orbits $\mathrm{Mon}\pi_1(\mathbb{P}_2^*)b(t), t \in U$, fill an open fundamental domain of $\mathbb{B}$ with respect to $\Gamma(\sqrt{-3})$.

2

4.5 Theorem ([H 86]). *The Baily-Borel compactification* $\widehat{\mathbb{B}/\Gamma(\sqrt{-3})}$ *of* $\mathbb{B}/\Gamma(\sqrt{-3})$ *is the projective plane* $\mathbb{P}^2$. *The locally finite analytic quotient map* $\mathbb{B} \to \mathbb{B}/\Gamma(\sqrt{-3})$ *is branched along* $\Delta \cap (\mathbb{B}/\Gamma(\sqrt{-3}))$. *The preimage of the branch locus is* $\Gamma\mathbb{D}$ *for* $\mathbb{D}$ *a subdise of* $\mathbb{B}$ *(e.g.* $z_1 = 0$*, if* z_1, z_2 *are the coordinates defining* $\mathbb{B}$ *by* $|z_1|^2 + |z_2|^2 < 1$*). The cusp point set* $\widehat{\mathbb{B}/\Gamma(\sqrt{-3})} \setminus \mathbb{B}/\Gamma(\sqrt{-3})$ *consists precisely of the triple points of* Δ.

2

Comparing with the Baily-Borel compactification of $\mathbb{B}/\mathrm{Mon}\,\pi_1(\mathbb{P}_2^*)$ we concluded with an argument of local branching that

4.6 Corollary ([H86]).

$$\mathrm{Mon}\pi_1(\mathbb{P}_2^*) = \mathbb{P}\Gamma(\sqrt{-3}).$$

2

Looking back to diagrams (3.21), (4.1) we get

4.7 Corollary.

$$\mathbb{P}\mathrm{Mon}\mathfrak{Z}_{\mathrm{col}} = \pi_1(\mathbb{P}_2^*) = \mathbb{P}\Gamma(\sqrt{-3})$$

and

$$\mathrm{Mon}\mathfrak{Z}_{\mathrm{col}} = \Gamma(\sqrt{-3}).$$

2

The factor group $\pi_1(\mathbb{P}_2^*/S_4)/\pi_1(\mathbb{P}_2^*)$ is S_4, see diagram (3.18). The middle part of diagram (4.1) translates this isomorphy to

$$\mathbb{P}\mathrm{Mon}\mathfrak{Z}/\mathbb{P}\mathrm{Mon}\mathfrak{Z}_{\mathrm{col}} \cong S_4, \text{ hence } \quad \mathrm{Mon}\mathfrak{Z}_{\mathrm{col}} \cong \Gamma(\sqrt{-3}),$$

see diagram (3.21). From the corollary and diagram (3.21) it follows that also

$$\Gamma/\mathrm{Mon}\mathfrak{Z}_{\mathrm{col}} = \Gamma/\Gamma(\sqrt{-3}) \cong S_4 \cong \mathrm{Mon}\mathfrak{Z}/\mathrm{Mon}\mathfrak{Z}_{\mathrm{col}}.$$

Since $\mathrm{Mon}\mathfrak{Z} \subseteq \Gamma$ it follows that $\Gamma = \mathrm{Mon}\mathfrak{Z}$, hence all inclusions in the first row of diagram (3.21) are equalities. Altogether we get what we want to prove:

4.8 Theorem. *With the above notations it holds that*

$$\mathrm{Mon}\mathfrak{Z} = \mathbb{S}p(6,\mathbb{Z})_{\mathrm{Pic}} = Sp(6,\mathbb{Z})_\alpha = \Gamma = \mathbb{U}((2,1),\mathfrak{O}).$$

2

4.9 Corollary. *The monodromy group* $Mon\mathfrak{Z} = \mathbb{U}((2,1),\mathfrak{O})$ *acts simply-transitive on the set of Picard bases of each smooth 3-sheeted Galois cover* C *of* $\mathbb{P}^1$ *of genus 3.*

Proof. The group $\mathbb{G}l_3(\mathfrak{O})$ acts simply transitive on the set of $\mathfrak{O}$-bases of $H_1(C,\mathbb{Z}) \cong \mathfrak{O}^3$. Therefore the action of $\Gamma \subset \mathbb{G}l_3(\mathfrak{O})$ must be simple.

It remains to prove that the action is transitive. Let $*\alpha, *\beta$ be two Picard bases on C. Since both are normal bases they are $Sp(6,\mathbb{Z})$-equivalent. Moreover, by definition of $Sp(6,\mathbb{Z})_{\mathrm{Pic}}$ they are equivalent with respect to the latter group. By the theorem the $\mathfrak{O}$-bases α, β are Γ-equivalent.

2

5. Moduli interpretations

There are two well-known moduli interpretations 5.1 und 5.2.

5.1 Theorem ([H86]). *The ball quotient surface* $\widehat{\mathbb{B}/\Gamma}$ *is the (compactified) moduli space of Picard curves.*

By Theorem 4.3 this moduli space is isomorphic to $\mathbb{P}^2/S_4$, and Theorem 4.5 teaches us that $\mathbb{P}^2/S_4 \cong \widehat{\mathbb{B}/\Gamma(\sqrt{-3})}/S_4 = \widehat{\mathbb{B}/\Gamma}$.

2

5.2. Theorem (Shimura [Shi], see also [B-L] for a more actual version). $\widehat{\mathbb{B}/\Gamma}$ *is the (compactified) moduli space of (principally) polarized abelian 3-folds with* K*-multiplication of signature* $(2,1)$.

2

By definition, an abelian variety A has K-multiplication, if there is an embedding of K into the endomorphism algebra $\mathrm{End}_{\mathbb{Q}}A = \mathbb{Q} \otimes \mathrm{End}\ A$. Signature $(2,1)$ means that the corresponding K-action on T_0A (tangent space at 0) can be diagonalized such that the restricted action on the diagonalizing lines appears twice as identical character of K^* and once as its conjugation. For more details about compatibility of polarization and K-multiplication according to the general concept of complex Shimura-varieties we refer to [Shi] or [B-H].

Theorem 5.1 extends now to the

5.3 Theorem. *The ball quotient surface* $\widehat{\mathbb{B}/\Gamma}$ *is the compactified moduli surface of all curves of genus 3 which are 3-sheeted Galois covers of* $\mathbb{P}^1$.

We have a *Shimura diagram* (in [H95] we called it *Schottly-Torelli diagram)*

$$\begin{array}{ccc} \mathbb{B} & \hookrightarrow & \mathbb{H}_3 \\ \downarrow & & \downarrow \\ \mathbb{B}/\Gamma & \longrightarrow & \mathcal{A}_3 \end{array} \tag{5.4}$$

extending

$$\begin{array}{cccc} & \mathbb{B}^* & \hookrightarrow & \mathbb{H}_3^* \\ & \downarrow & & \downarrow \\ (\mathbb{P}^2 \setminus \Delta)/S_4 = & \mathbb{B}^*/\Gamma & \hookrightarrow & \mathcal{A}_3^* \end{array} \tag{5.4*}$$

where $\mathbb{B}^*$ denotes the preimage of $\mathbb{P}^2 \smallsetminus \Delta$ along the quotient map $\mathbb{B} \to \mathbb{B}/\Gamma(\sqrt{-3}) = \mathbb{P}^2 \smallsetminus \{$ 4 points$\}$, and $\mathbb{H}_3^*$ is the preimage of $\mathcal{A}_3^* \subset \mathcal{A}_3$ along the quotient morphism $\mathbb{H}_3 \to \mathbb{H}_3/Sp(6,\mathbb{Z}) = \mathcal{A}_3$ (look at the end of section 1). The points of $\mathcal{A}_3^*$ correspond to Jacobians of smooth curves of genus 3, and the points of $\mathbb{B}^*/\Gamma$ correspond to smooth Picard curves.

By Torelli's theorem the correspondence of smooth Picard curves to their Jacobians defines the embedding $\mathbb{B}^*/\Gamma \hookrightarrow \mathcal{A}_3^*$, which is algebraic because of a theorem of Chow: Namely, after compactification and a suitable singularity resolution $\widetilde{\mathbb{B}/\Gamma}$ of $\widehat{\mathbb{B}/\Gamma}$, not changing $\mathbb{B}^*/\Gamma$, one gets an analytic morphism $\widetilde{\mathbb{B}/\Gamma} \to \hat{\mathcal{A}}_3$ which has to be algebraic by Chow. Furthermore, $\mathbb{B}^*/\Gamma$ is obviously a Zariski-open subset of $\widetilde{\mathbb{B}/\Gamma}$.

The lower arrow in diagram (5.4) represents a rational morphism extending the corresponding embedding of $(5.4)^*$. Namely, the quotient map on the left side of (5.4) comes from the restriction of the $Sp(6,\mathbb{Z})$-action on $\mathbb{H}_3$ to the $\Gamma = Sp(6,\mathbb{Z})_{\mathrm{Pic}}$-action on $\mathbb{B}$.

Theorem 5.3 has to be understood in the following sense:

5.4 Proposition. (i) *The period matrices of all smooth 3-sheeted Galois covers of $\mathbb{P}^1$ of genus 3 correspond precisely to the points of $\mathbb{B}^*$.*
(ii) *The moduli points of the same curves fill precisely* $(\mathbb{P}^2 \smallsetminus \Delta)/S_4 = \mathbb{B}^*/\Gamma$.

Proof. By Proposition 2.5 the image points of the period matrices (with respect to the bijective correspondence (2.6)) belong to $\mathbb{B} \subset \mathbb{H}_3$. If $\mathbb{B}^\#$ denotes the image, then we know that

$$\mathbb{B}^* \subseteq \mathbb{B}^\# \subseteq \mathbb{B} \subset \mathbb{H}_3$$

because the period matrices of smooth Picard curves fill $\mathbb{B}^*$. Namely, the moduli points of smooth Picard curves fill precisely $(\mathbb{P}^2 \smallsetminus \Delta)/S_4$, and the Picard period matrices of one of these curves C fill precisely an orbit $\Gamma b = Sp(6,\mathbb{Z})_{\mathrm{Pic}} b$, $b \in \mathbb{B}$ suitable. Since each smooth Picard curve $Y^3 = P_4(X)$ belongs to the curve class of Proposition (5.4) we get $\mathbb{B}^* \subseteq \mathbb{B}^\#$.

Now it suffices to check that the set $M^\#$ of moduli points of our curve class is a subset of $(\mathbb{P}^2 \smallsetminus \Delta)/S_4$. This means that each smooth 3-sheeted Galois cover C of $\mathbb{P}^1$ of genus 3 is isomorphic to a smooth Picard curve. This is easy to see: Let $\mathbb{C}(C)$ be the function field of C, $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(X)$. By Kummer theory the cyclic field extension $\mathbb{C}(C)/\mathbb{C}(X)$ is generated by a third root of an element of $\mathbb{C}(X)$, say

$$\mathbb{C}(C) = \mathbb{C}(X)(y), \quad y^3 = \frac{f(X)}{g(X)}, \quad f(X), g(X) \in \mathbb{C}[X].$$

Multiplying by $g^3(X)$ and changing y by $yg(X)$ it is justified to assume that $g(X) = 1$, that means

$$y^3 = f(X), \quad f(X) \in \mathbb{C}[X].$$

In [K-K] it is proved that one can choose more precisely

$$y^3 = (X-a_1)(X-a_2)(X-a_3)(X-a_4)(X-a_5)^2 =: f(X) \tag{5.5}$$

as affine equation for a model of C. As described in [Sha], I, §3.1, one finds a Picard equation by means of a birational transformation in the following manner: multiply (5.6) by $(X-a_5)^{-6}$, substitute $\frac{y}{(X-a_5)^2}$ by U and $(X-a_5)^{-1}$ by V. Then $\frac{f(X)}{(X-a_5)^4} = \frac{F(X-a_5)}{(X-a_5)^4}$ is a polynomial $p_4(V)$ of degree 4 and $U^3 = p_4(V)$ is the equation we look for.

2

As corollary we get the following result of Schottky-type

5.6 Proposition. *A matrix* $\Pi = (\Pi_1|\Pi_2) \in Mat_{3\times 6}(\mathbb{C})$ *is the period matrix of a smooth 3-sheeted Galois cover of* $\mathbb{P}^1$ *of genue 3 if and only if it is* $\mathbb{G}l_3(\mathbb{C})$*-equivalent (by left multiplication) to a Picard matrix* Π' *(defined in 2.6) such that the image of* Π' *along the bijection (2.3) belongs to* $\mathbb{B}^*$*.*

2

5.7 Remark. In [K-K] the authors prove that smooth genus 3 curves with automorphism group $\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/6\mathbb{Z}$ are precisely those which have a smooth Picard curve model. So these isomorphy classes are precisely represented by $\mathbb{P}_2^*/S_4$. The moduli points of these curves with automorphism group $\mathbb{Z}/6\mathbb{Z}$ correspond precisely to the S_4-orbit of the three lines on $\mathbb{P}^2$ going to pairs of the three double points of Δ. The corresponding Picard curves are of equation type $Y^3 = X^4 + aX^2 + b$, see [K-K].

5.8 Remark. The curves represented by $\mathbb{P}_2^*/S_4$ are not hyperelliptic. This is clear for the curves with automorphism group $\mathbb{Z}/3\mathbb{Z}$ because hyperelliptic curves have an automorphism of order 2. For the smooth curves C of equation type $Y^3 = X^4 + aX^2 + b$ the automorphism group is generated by $(x, y) \mapsto (-x, \varrho y)$. The quotient of C by the subgroup of order 2 is obviously an elliptic curve $E : y^3 = U^2 + aU + b$. Therefore C cannot be a 2-sheeted covering of $\mathbb{P}^1$. In [H86] we proved that the Picard curves corresponding to smooth points of Δ are singular models of a smooth hyperelliptic curve of genus 2.

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